3.) Eikonal Theory and Ray Equations

Special Topic – Eikonal Theory

→ Eikonal Theory and Mechanics of Wave Propagation

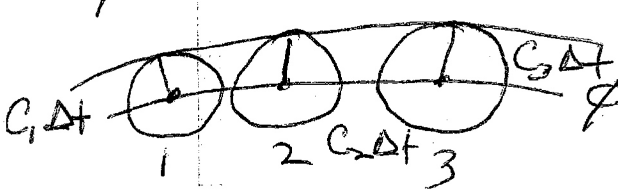
Here, seek to provide description of wave propagation via mechanics. Description is in terms of rays – paths followed by wave – and valid in short wavelength limit. How short?

i) Consider Helmholtz eqn.

$$\nabla^2\psi + \frac{\omega^2}{c^2(\underline{x})}\psi = 0 \qquad c^{-2}(\underline{x}) = \frac{n^2(\underline{x})}{c^2}$$

index refraction

ii) Now, consider wavefront described by $\phi = \text{const}$



$\phi = \text{const}$ at $t_0 + \Delta t$

$\phi(\underline{x}) = \text{const.}$ at t_0

Huygens' Construction

iii) each point on ϕ surface emits disturbance, propagating at local c, i.e.

envelope of such spheres is wave-front at $t_0 + \Delta t$

$\therefore$ rays orthogonal to wave fronts.

Huygen's construction ↔ process of evolving envelope via following spherical emissions.

Now, infinitesimal displacement vector along light ray $\equiv d\underline{\Gamma}$ i.e. add'l ds.

then

$$\underline{\nabla}\phi \cdot d\underline{\Gamma} = \omega dt$$

net phase advance

or, as $\underline{\nabla}\phi \parallel d\underline{\Gamma}$

$$|\nabla\phi|\,|d\Gamma| = \omega dt$$

but $dt = |d\Gamma|/c$, by definition

$$\therefore \quad |\underline{\nabla}\phi|\,\cancel{|d\Gamma|} = \frac{\omega}{c}\,\cancel{|d\Gamma|}$$

$$\Rightarrow \boxed{\left[(\partial_x\phi)^2 + (\partial_y\phi)^2 + (\partial_z\phi)^2\right] = \omega^2/c^2}$$

— eikonal equation.

More directly, Helmholtz equation $\Rightarrow$

$$\nabla^2\psi + \frac{\omega^2}{c^2(\underline{x})}\psi = 0$$

phase – fastest dependence (assumed)

$$\psi(\underline{x}) = A e^{i\phi(\underline{x})}$$

$$\Rightarrow \left[-|\underline{\nabla}\phi|^2 A + i \nabla^2\phi A + i \underline{\nabla} A \cdot \nabla\phi + \dots \right] e^{i\phi} = -\frac{\omega^2}{c(\underline{x})^2} A e^{i\phi}$$

$$\therefore \quad \boxed{|\nabla\phi|^2 = \omega^2 / c(\underline{x})^2}$$

eikonal equation.

Now $\underline{\nabla}\phi \cdot d\underline{x} \equiv$ net phase increment

so clearly $\underline{\nabla}\phi = \underline{k} = \underline{k}(x)$ (in WKB sense)

$$\Rightarrow \quad \phi = \int \underline{k} \cdot dx = \int \underline{\nabla}\phi \cdot d\underline{x}$$

so $\psi = A e^{i\left[\int \underline{k} \cdot d\underline{x} - \omega t\right]}$ is wave function.

Now, can combine $\underline{x}$, t evolution so

$$\Phi \equiv \int \underline{k} \cdot d\underline{x} - \omega t$$

$$\therefore \quad \underline{k} = \nabla \Phi, \qquad \omega = -\frac{\partial \Phi}{\partial t}$$

⇒ Now, seek equations which evolve ray path in time, i.e. which give

$$\left.\begin{array}{ll} \text{ray position} & \underline{x} \\ \text{ray direction} & \underline{k} \end{array}\right\} \text{ as function of time}$$

⇒ mechanics problem!

a.) poor man's version.

For linear wave, $\omega = \text{const.}$

Since: $\omega = \omega(\underline{k}, \underline{x})$ ⇒

$$\frac{d\omega}{dt} = 0 = \cancel{\frac{\partial \omega}{\partial t}} + \frac{\partial \omega}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt}$$

$$\frac{\partial \omega}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} \quad \Rightarrow \quad \boxed{\begin{aligned} \frac{d\underline{k}}{dt} &= -\frac{\partial \omega}{\partial \underline{x}} \\ \frac{d\underline{x}}{dt} &= \frac{\partial}{\partial \underline{k}} \omega \end{aligned}}$$

eikonal eqns.

With, of course: $\omega^2 = c(\underline{x})^2 k^2$

$$\Rightarrow \quad \cancel{2}\omega\, \partial\omega = \cancel{2}\underline{k}\cdot d\underline{k}\; c^2(\underline{x})$$

$$\partial\omega = \hat{\underline{k}}\cdot d\underline{k}\; c(\underline{x})$$

$$\underline{k} = k\,\hat{\underline{k}}$$

$$\frac{\partial\omega}{\partial\underline{k}} = c(\underline{x})\,\hat{\underline{k}} \qquad \hat{\underline{k}} = \frac{\nabla\phi}{|\nabla\phi|}$$

$\equiv$ group velocity

and

$$\frac{\partial\omega}{\partial\underline{x}} = \frac{\partial}{\partial\underline{x}}\left[c(\underline{x})^2 k^2\right]^{1/2}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{d\underline{x}}{dt} &= c(\underline{x})\,\hat{\underline{k}} \\ \frac{d}{dt}\underline{k} &= -\frac{\partial}{\partial\underline{x}}\left[c(\underline{x})^2 k^2\right]^{1/2} \end{aligned}}$$

More rigorously:

$$\Phi = \int \underline{k}\cdot d\underline{x} - \omega t$$

$$d\Phi = \underline{k}\cdot d\underline{x} - \omega\, dt$$

$$\underline{\Phi} = \int_{\underline{x}_0,\, \underline{k}_0,\, t_0}^{\underline{x}_1,\, \underline{k}_1,\, t_1} \left[\underline{k} \cdot d\underline{x} - \omega\, dt \right]$$

Now, assert ray will follow path which extremizes $\underline{\Phi}$, i.e. $\left\{ \begin{array}{l} \text{maximizes accumulated} \\ \text{phase.} \end{array} \right.$

$$\delta \underline{\Phi} = \delta \int \left[\underline{k} \cdot d\underline{x} - \omega\, dt \right] = 0$$

$$= \int \left[\delta \underline{k} \cdot d\underline{x} + \underline{k} \cdot d\, \delta \underline{x} \right] - \int \left(\frac{\partial \omega}{\partial \underline{k}} \cdot \delta \underline{k} + \frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} \right) dt$$

as $\delta \underline{x} = \delta \underline{k} = 0$ at end points,

$$\delta \underline{\Phi} = 0 = \int \left[\delta \underline{k} \cdot d\underline{x} - d\underline{k} \cdot \delta \underline{x} \right] - \int \left[\left(\frac{\partial \omega}{\partial \underline{k}} \cdot \delta \underline{k} \right) dt + \left(\frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} \right) dt \right]$$

$\left\{ \begin{array}{l} \text{a la} \\ \text{prin. least} \\ \text{action} \end{array} \right.$

$$d\underline{x} = \left(\frac{\partial \omega}{\partial \underline{k}} \right) dt \qquad \Rightarrow \qquad \frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}}$$

$$d\underline{k} = -\frac{\partial \omega}{\partial \underline{x}}\, dt \qquad \qquad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

$\Big\}$ eikonal eqns.

Now, eikonal equations:

→ evolve ray in $\underline{x}, \underline{k}$ phase space

→ Hamilton equations for ray in $(\underline{x}, \underline{k})$ phase space,

i.e. $\underline{x}, \underline{k}$ symmetrically treated

$$\frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} + \frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} = 0$$

$$\frac{\partial}{\partial \underline{x}} \cdot \frac{\partial \omega}{\partial \underline{k}} + \frac{\partial}{\partial \underline{k}} \left(-\frac{\partial \omega}{\partial \underline{x}} \right) = 0 \quad \checkmark$$

→ as $\dot{\underline{x}}, \dot{\underline{k}}$ equations Hamiltonian,

can define $\rho(\underline{x}, \underline{k}, t) \equiv$ wave (phonon) density in phase space)

$\sim N(\underline{x}, \underline{k}, t)$ ~ intensity

and use Liouville's Thm:

$$\frac{\partial \rho}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial \rho}{\partial \underline{x}} + \frac{d\underline{k}}{dt} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0$$

$$\frac{d\underline{x}}{dt} = \underline{v}_{gr} \quad , \quad \frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{v}_g \cdot \frac{\partial \rho}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0}$$

"Transport equation" for waves, i.e.

- gives intensity evolution, via ρ
- applications in radiation hydro., quasi-particle evolution, etc.

→ obvious analogy:

Particles	Rays
H	ω
$\underline{k}$	$\underline{p}$
$\underline{x}$	$\underline{q}$
S	ϕ

etc.

N.B. : Hamilton-Jacobi Eqn. is eikonal equation for Schroedinger Eqn.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

$$\psi = \psi_0 e^{iS/\hbar} \quad \Rightarrow \quad S = S(\underline{x}, t)$$

$$-\frac{\partial S}{\partial t} = \frac{1}{2m} (\nabla S)^2 + V$$

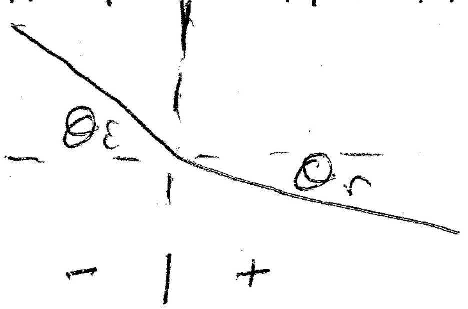
$$\Rightarrow \quad \frac{\partial S}{\partial t} + H(\nabla S) = 0$$

$\Rightarrow$ Classical trajectory as path of least phase accumulation in quantum/wave mechanical system.

Applications:

i.) Snells Law

$n = 1 \qquad n = n(\underline{x}) = n_1 > 1$



$$\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}} \Rightarrow \frac{dk_y}{dt} = 0$$

$$k_{y-} = k_{y+} \Rightarrow$$

$$k_- \sin\theta_i = k_+ \sin\theta_r$$

but $\quad k_+ = n_1 k_- \qquad\qquad k_+^2 = n^2 \frac{\omega^2}{c^2}$

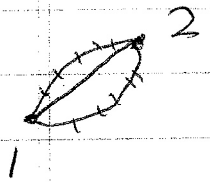
$$\sin\theta_i = n \sin\theta_r$$

more generally, $\quad n_0 \sin\theta_i = n_1 \sin\theta_r.$

Ray Theory Revisited

→ Fermat's Principle

Fermat's Principle ⟺ Ray will traverse path of minimal *time* of propagation



Path actually followed is one of minimal *time*.

so, if $T = \int_{\underline{x}_1}^{\underline{x}_2} \frac{d\ell}{c(\underline{x})}$ $\qquad$ $c(\underline{x}) = \frac{c_0}{n(\underline{x})} \rightarrow$ indx

then $\delta T = 0$. $\qquad$ $T = \int_{\underline{x}_1}^{\underline{x}_2} d\ell \, n(\underline{x})$

example:

i.e. $x_1 = (x_1, y_1)$

$x_2 = (x_2, y_2)$

$d\ell = ds$ $\qquad$ $ds^2 = dx^2 + dy^2$

$c^2 = c^2(y) = c_0^2 / n^2(y)$

$$\therefore \quad T = \int_{\underline{x}_1}^{\underline{x}_2} d\ell \, n(\underline{x}) = \int_{\underline{x}_1}^{x_2} dx \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} n(y)$$

⇒ can derive equation for path $y(x)$

$$\delta T = \int_{x_1}^{x_2} dx \left\{ \frac{\partial n}{\partial y} \left(1+y'^2\right)^{1/2} \delta y + \frac{y'}{\left(1+y'^2\right)^{1/2}} n(y)\, \delta y' \right\}$$

IBP ⇒

$$= \int_{x_1}^{x_2} dx \left\{ \frac{\partial n}{\partial y} \left(1+y'^2\right)^{1/2} - \frac{d}{dx}\left(\frac{y' n(y)}{\left(1+y'^2\right)^{1/2}} \right) \right\} \delta y$$

$$\delta T = 0 \Rightarrow \boxed{ \frac{d}{dx}\left(\frac{n(y)\, y'}{\left(1+y'^2\right)^{1/2}} \right) - \left(1+y'^2\right)^{1/2} \frac{\partial n}{\partial y} = 0. }$$

→ Eqn. for Ray Path

→ Analogue (loosely) of Lagrange's Equation.

Alternative Derivation – Eikonal Theory

For path, use analogue of abbreviated action

$$S_0 = \int \underline{p} \cdot d\underline{q}$$

so

$$\Phi_0 = \int \underline{k} \cdot d\underline{x}$$

"abbreviated phase"

Now, eikonal equation $\Rightarrow$

$$(\nabla \phi)^2 = \frac{\omega^2}{c^2(\underline{x})} = \frac{\omega^2 n^2(\underline{x})}{c_0^2}$$

$$|\nabla \phi| = \frac{\omega}{c_0} n(\underline{x})$$

$$\underline{k} \cdot d\underline{x} = |\nabla \phi| \, d\ell \quad \Rightarrow$$

$$\Phi_0 = \frac{\omega}{c_0} \int_{\underline{x}_1}^{\underline{x}_2} n(\underline{x}) \, d\ell$$

$\delta \Phi_0 = 0 \quad \Rightarrow$ Fermat's Principle.

$\therefore$ have established equivalence between Fermat's principle and Principle of 'Minimal Phase Accumulation'

General equation for ray :

$$\delta \int_{\underline{x}_1}^{\underline{x}_2} n(\underline{x})\, d\ell = 0$$

$$\delta \int_{\underline{x}_1}^{\underline{x}_2} n(\underline{x})\, d\ell = \int_{\underline{x}_1}^{\underline{x}_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x}\, d\ell + n(\underline{x})\, \delta d\ell \right\}$$

now $\quad d\ell^2 = d\underline{x} \cdot d\underline{x}$

$$d\ell\, \delta d\ell = d\underline{x} \cdot d\delta\underline{x}$$

$$\delta d\ell = \frac{d\underline{x}}{d\ell} \cdot d\delta\underline{x}$$

$$\delta \int_{\underline{x}_1}^{\underline{x}_2} n(\underline{x})\, d\ell = \int_{\underline{x}_1}^{\underline{x}_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x}\, d\ell + n(\underline{x}) \frac{d\underline{x}}{d\ell} \cdot d\delta\underline{x} \right\}$$

IBP $\Rightarrow$

$$= n(\underline{x}) \frac{d\underline{x}}{d\ell} \cdot \delta\underline{x} \Bigg|_{\underline{x}_1}^{\underline{x}_2} + \int_{\underline{x}_1}^{\underline{x}_2} \left\{ \frac{\partial n}{\partial \underline{x}} \cdot \delta \underline{x}\, d\ell - \frac{d}{d\ell}\left(n(\underline{x}) \frac{d\underline{x}}{d\ell} \right) \cdot \delta\underline{x}\, d\ell \right\}$$

$$\delta \Phi_0 = \cancel{ept} + \int_{\underline{x}_1}^{\underline{x}_2} \left\{ \frac{\partial n}{\partial \underline{x}} - \frac{d}{dl}\left(n(\underline{x}) \frac{d\underline{x}}{dl} \right) \right\} \cdot \delta \underline{x} \, dl$$

$$\Rightarrow \boxed{\frac{d}{dl}\left(n(\underline{x}) \frac{d\underline{x}}{dl} \right) - \frac{\partial n}{\partial \underline{x}} = 0}$$

Ray path equation

observe can re-write:

$$n(\underline{x}) \frac{d^2 \underline{x}}{dl^2} = \frac{\partial n}{\partial \underline{x}} - \left(\frac{\partial n}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dl} \right) \frac{d\underline{x}}{dl}$$

$$\boxed{\frac{d^2 \underline{x}}{dl^2} = \frac{1}{n(\underline{x})} \frac{\partial n}{\partial \underline{x}} - \frac{1}{n(\underline{x})} \left(\frac{\partial n}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dl} \right) \frac{d\underline{x}}{dl}}$$

Meaning?

→ $d\underline{x}/dl$ is unit tangent to ray

$$dl \, dl = d\underline{x} \cdot d\underline{x}$$

$$\underline{t} = d\underline{x}/dl$$

so

→ $\frac{d^2 \underline{x}}{d\ell^2}$ corresponds to ray curvature $\underline{K}$

$1/|\underline{K}| \equiv$ effective radius of curvature.

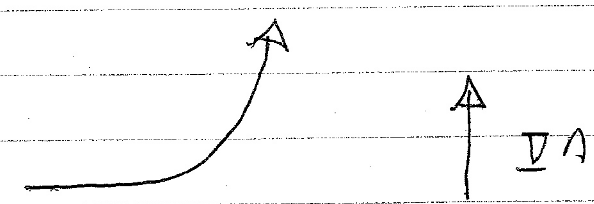
so

$$\underline{K} = \frac{1}{n}\underline{\nabla} n - \frac{1}{n}\left(\underline{\nabla} n \cdot \underline{t}\right)\underline{t}$$

$$= \frac{1}{n}\,\underline{\nabla} n \cdot \hat{n}$$

unit vector normal to path

Loosely put: Ray curves toward region of increasing index



Now, could ask: how does ray envelope function evolve along ray.

Analogue: Particles $\leftrightarrow$ Rays

$$S \leftrightarrow \Phi$$

so, question analogous to determining $S(q, t)$!

Recall:

allow vary, now

$$\delta\Phi = \delta\int_{\underline{x}_1}^{\underline{x}} \frac{\omega}{c_0} n(x)\, d\ell$$

$\underline{x}_1$ ↳ fixed

$$= \frac{\omega}{c_0} n(\underline{x}) \frac{d\underline{x}}{d\ell} \cdot d\underline{x} \Big|_{\underline{x}_1}^{\underline{x}}$$

$$+ \int_{\underline{x}_1}^{\underline{x}} d\ell\, d\underline{x} \cdot \left\{ \frac{\partial n}{\partial \underline{x}} - \frac{d}{d\ell}\left(n(\underline{x}) \frac{d\underline{x}}{d\ell} \right) \right\}$$

As ray path satisfies $\{\ \} = 0$,

$$\boxed{\frac{\partial \Phi}{\partial \underline{x}} = \frac{\omega}{c_0} n(\underline{x}) \frac{d\underline{x}}{d\ell}}$$

$\Big\{$ Equation for Envelope/Phase Function

Observe: Path and Phase Equations $\Rightarrow$

$$\left.\begin{aligned} \underline{p} &\equiv \frac{\omega}{c_0} n(\underline{x}) \frac{d\underline{x}}{d\ell} \\ \frac{d\underline{p}}{d\ell} &= \frac{\omega}{c_0} \underline{\nabla} n(\underline{x}) \end{aligned}\right\} \text{Equations with Canonical Structure.}$$

Note obvious analogy with particles:

Particles	Rays/Paths
S_0	Φ_0
$\delta \int \sqrt{2m(E-V)} = 0$	$\delta \int \frac{\omega}{c_0} n \, d\ell = 0$
$\sqrt{E-V} \frac{d}{d\ell}\left[\sqrt{E-V} \frac{d\underline{x}}{d\ell}\right] = -\partial \underline{V}/\partial r$	$\frac{d}{d\ell}\left(\frac{\omega}{c_0} n(\underline{x}) \frac{d\underline{x}}{d\ell}\right) = \frac{\omega}{c_0} \underline{\nabla} n$